MATRIX POWERS

1.) In this paper, the powers of a matrix are investigated. We start with the matrix M given by:

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Taking the square of this matrix means just multiplying M by itself:

$$M^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

In order to simplify this equation, basic matrix multiplication is used. To multiply two general matrices M_1 and M_2, where

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

the general formula is:

$$M_1 M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

In our case a = e = d = h = 2, b = f = c = g = 0. Substituting these values into the matrix above,

$$M_1 M_2 = \begin{pmatrix} (2)(2) + 0 & 0 + 0 \\ 0 + 0 & 0 + (2)(2) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

so that the square of the matrix M becomes

$$M^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 2^2 \end{pmatrix}$$

The last equality takes note of the fact that the resulting matrix has elements that are a power of the elements of the original matrix M: in this case, $2^2 = 2 * 2 = 4$. This is a pattern that continues when the matrix M is taken to higher powers:

$$M^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^3 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 2^3 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 2^4 & 0 \\ 0 & 2^4 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^5 = \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix} = \begin{pmatrix} 2^5 & 0 \\ 0 & 2^5 \end{pmatrix}$$

To find M to the 10th power, two M^5 matrices can be multiplied together:

$M^{10} = M^5 * M^5$

which uses the fact that $M^n * M^m = M^{n+m}$

$$M^{10} = M^5 M^5 = \begin{pmatrix} 2^5 & 0 \\ 0 & 2^5 \end{pmatrix}\begin{pmatrix} 2^5 & 0 \\ 0 & 2^5 \end{pmatrix} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}$$

$$M^{20} = M^{10} M^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} = \begin{pmatrix} 2^{20} & 0 \\ 0 & 2^{20} \end{pmatrix}$$

$$M^{50} = M^{10} M^{20} M^{20} = M^{10}\begin{pmatrix} 2^{20} & 0 \\ 0 & 2^{20} \end{pmatrix}\begin{pmatrix} 2^{20} & 0 \\ 0 & 2^{20} \end{pmatrix} = M^{10}\begin{pmatrix} 2^{40} & 0 \\ 0 & 2^{40} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 2^{40} & 0 \\ 0 & 2^{40} \end{pmatrix} = \begin{pmatrix} 2^{50} & 0 \\ 0 & 2^{50} \end{pmatrix}$$

From all of these matrix products a pattern emerges, that the matrix M taken to the power n is in general:

$$M^n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$$

so that the power n of the matrix M has elements that are taken to the power of n.

2.) Considering now the matrices

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = 2\begin{pmatrix} (3-1)^2+1 & (3-1)^2-1 \\ (3-1)^2-1 & (3-1)^2+1 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 2\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 2\begin{pmatrix} 15+3 & 5+9 \\ 9+5 & 3+15 \end{pmatrix} = 2*2\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}$$

$$= 2^{3-1}\begin{pmatrix} (3-1)^3+1 & (3-1)^3-1 \\ (3-1)^3-1 & (3-1)^3+1 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^3\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 2^2\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 2^2\begin{pmatrix} 27+7 & 9+21 \\ 21+9 & 7+27 \end{pmatrix} = 2^3\begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix}$$

$$= 2^{4-1}\begin{pmatrix} (3-1)^4+1 & (3-1)^4-1 \\ (3-1)^4-1 & (3-1)^4+1 \end{pmatrix}$$

The pattern that is emerging is that

$$P^n = 2^{n-1}\begin{pmatrix} (3-1)^n+1 & (3-1)^n-1 \\ (3-1)^n-1 & (3-1)^n+1 \end{pmatrix}$$

Considering the second part of the problem:

$$S = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^2$$

$$= \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

$$= 2^{2-1}\begin{pmatrix} (4-1)^2+1 & (4-1)^2-1 \\ (4-1)^2-1 & (4-1)^2+1 \end{pmatrix}$$

$$S^3 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^3$$

$$= 2\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = 2*2\begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix}$$

$$= 2^{3-1}\begin{pmatrix} (4-1)^3+1 & (4-1)^3-1 \\ (4-1)^3-1 & (4-1)^3+1 \end{pmatrix}$$

so that the pattern emerging here is:

$$S^n = 2^{n-1}\begin{pmatrix} (4-1)^n+1 & (4-1)^n-1 \\ (4-1)^n-1 & (4-1)^n+1 \end{pmatrix}$$

3.) For matrices of the form

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

the k = 1 matrix is the matrix M of part (1). For k = 2, the matrix P of part(2) results, and for k=3, the matrix S of part (2) results. For all of these matrices, the following relation between k and n holds:

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

for k and n that are positive integers: k = 1, 2, 3... and n = 1, 2, 3...

Using k = 0,

$$T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2^2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T^4 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2^2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2^3\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

so that the pattern that emerges is

$$T^n = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

for k = 0 (which predicts what the general equation found, so k can be 0 for the general equation). For k= -1,

$$U = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

$$U^2 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 2\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$U^3 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = 2\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} = 2*2\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

$$U^4 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^3 \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = 2^2\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = 2^3\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$U^5 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^4 \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = 2^3\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = 2^3\begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} = 2^4\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

so that the pattern that emerges is:

$$U^n = 2^{n-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for n = even number, and

$$U^n = 2^{n-1}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

for n = odd number. This is what the general equation predicts, which shows that k can be a negative integer.

The general equation found for k and n:

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

holds only for k equal to any integer: k = ...-3, -2, -1, 0,1, 2, 3...

4.)Technology to see what happens for other values of k and n. State the scope/limitations on k and n.
Using a TI-85, the matrices were calculated for various values of k and n. For instance, the M matrix of part (1), taken for negative values of n (negative powers) became
M^{-1}= 1/4 M
M^{-2}= 1/8 M
this shows a pattern of
$M^{n}= 2^{n-1}$ M
for n = -1, -2, -3... For the M matrix, k = 1, so the general equation becomes

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for k = 1. This shows that n can be a negative integer for the matrix M.

$$P^{-1} = \begin{pmatrix} .375 & -.125 \\ -.125 & .375 \end{pmatrix}$$

If the formula

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

held, then (for P, k=2):

$$P^{-1} = 2^{-2}\begin{pmatrix} 2^{-1}+1 & 2^{-1}-1 \\ 2^{-1}-1 & 2^{-1}+1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1.5 & -.5 \\ -.5 & 1.5 \end{pmatrix} = \begin{pmatrix} .375 & -125 \\ -.125 & .375 \end{pmatrix}$$

so this shows that the general formula holds for negative n. For n = 0, the general equation predicts:

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The limitations on k and n are that k must be any integer:
k = ...-3, -2, -1, 0, 1, 2, 3....
and n is any integer:
n = ...-3, -2, -1, 0, 1, 2, 3...
for the general equation

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

to hold.

5.) To show why these results hold in general, consider the general matrix,

$$Z = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

and its first few squares:

$$Z^2 = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} = \begin{pmatrix} (k+1)^2+(k-1)^2 & 2(k+1)(k-1) \\ 2(k+1)(k-1) & (k+1)^2+(k-1)^2 \end{pmatrix}$$

$$= 2\begin{pmatrix} k^2+1 & k^2-1 \\ k^2-1 & k^2+1 \end{pmatrix}$$

$$Z^3 = 2\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}\begin{pmatrix} k^2+1 & k^2-1 \\ k^2-1 & k^2+1 \end{pmatrix}$$

$$= 2\begin{pmatrix} (k^2+1)(k+1)+(k-1)(k^2-1) & (k^2-1)(k+1)+(k^2+1)(k-1) \\ (k^2+1)(k-1)+(k^2-1)(k+1) & (k^2+1)(k+1)+(k-1)(k^2-1) \end{pmatrix}$$

$$= 2^2\begin{pmatrix} k^3+1 & k^3-1) \\ k^3-1 & k^3+1 \end{pmatrix}$$

$$Z^4 = 2^2\begin{pmatrix} k^3+1 & k^3-1) \\ k^3-1 & k^3+1 \end{pmatrix}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

$$= 2^3\begin{pmatrix} k^4+1 & k^4-1 \\ k^4-1 & k^4+1 \end{pmatrix}$$

This is showing the pattern:

$$Z^n = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

where k is any integer: k = ...-3, -2, -1, 0, 1, 2, 3... and n is any integer: n=...-3, -2, -1, 0, 1,2,3...

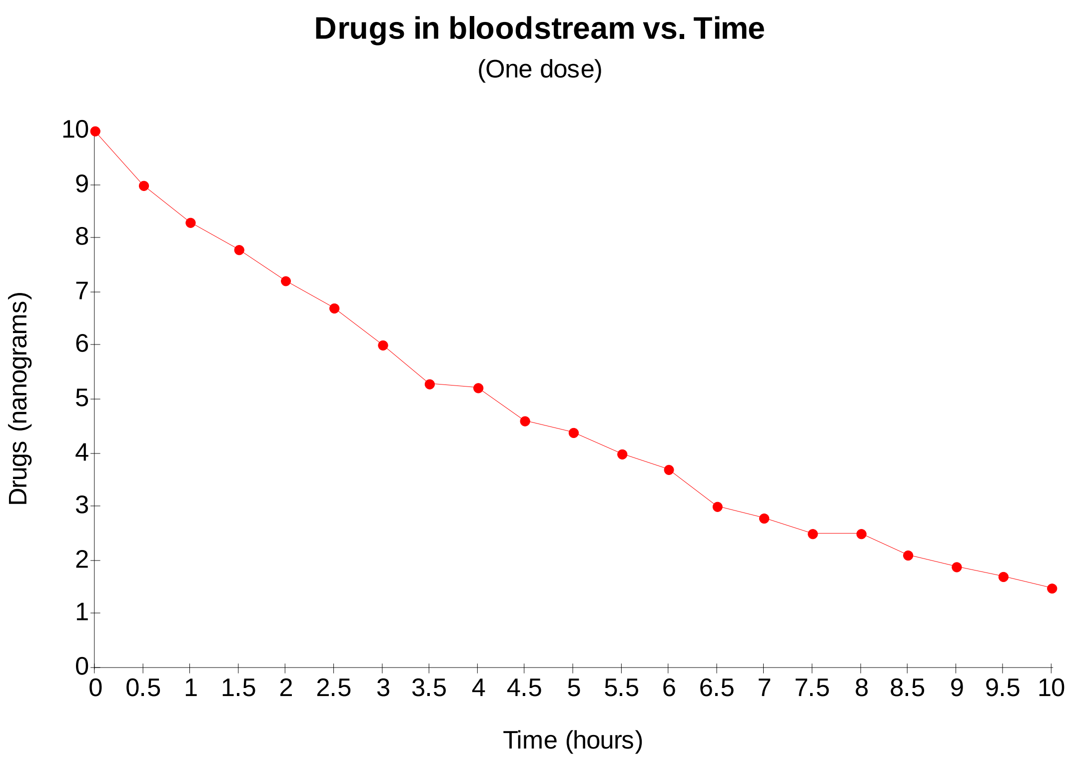
Modelling the Amount of a Drug in the Bloodstream

PART A

1.)A function to model this data:

0	10			
0.5	9			
1	8.3			
1.5	7.8			
2	7.2			
2.5	6.7			
3	6			
3.5	5.3			
4	5.2			
4.5	4.6			
5	4.4			
5.5	4			
6	3.7			
6.5	3			
7	2.8			
7.5	2.5			
8	2.5			
8.5	2.1			
9	1.9			
9.5	1.7			
10	1.5			

where the first column is the hour and the second column is the drugs in the bloodstream (in nanograms). This data is graphed below (using a spreadsheet):



PART B

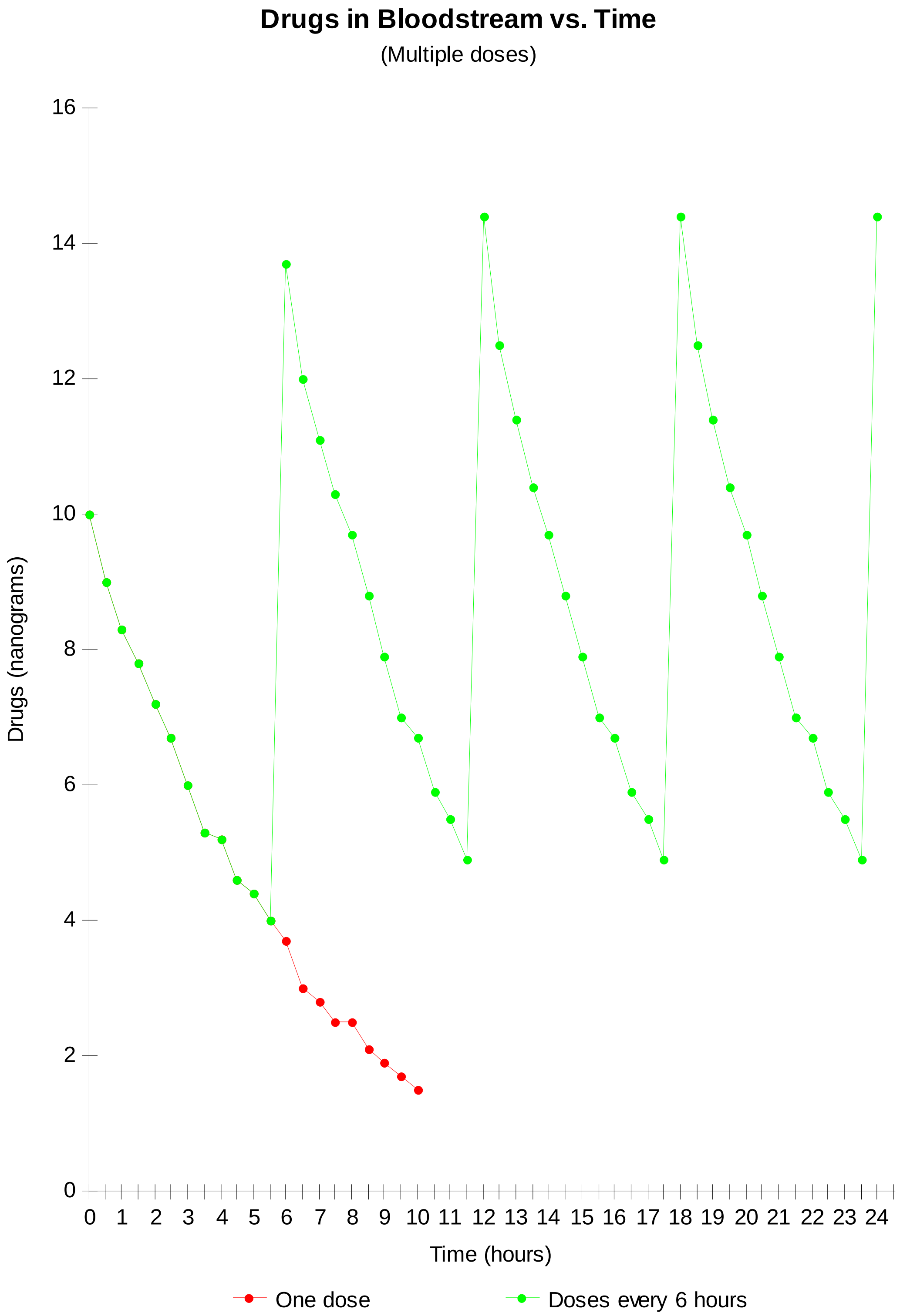
1.&2.)If the patient is administered the drug every 6 hours, then the amount of drug in the patient's bloodstream at hour 6 is assumed to be that of the graph of part A from hour 6 to the right + the graph to the left starting at 0 hours.

Example of the pattern used:

(drugs at hour 6 for part B) = (drugs at hour 6 of part A) + (drugs at hour 0 of part A)

(drugs at hour 6.5 for part B) = (drugs at hour 6.5 for part A) + (drugs at hour 0.5 of part A)

In this graph it is assumed that the amount of the drug in the bloodstream will decrease at the same rate, past the 10th hour of the part A graph. For 2 hours on the right hand side of the graph the amount of drugs decreased by 0.2 nanograms every 0.5 hours; it is assumed this rate is continued down to 0, and is added to the amount of drugs in the bloodstream from the other doses in the graph.



Ошибка! Раздел не указан.
Data for the multiple doses graph.
3.)The maximum amount of drugs in the bloodstream from doses every 6 hours = 14.4

nanograms. The minimum amount of drugs in the bloodstream from doses every 6 hours = 4 nanograms, which occurs just before the second dose. After the second dose is administered, the minimum amount in the bloodstream = 4.9 nanograms.

4a) If no further doses are taken, the amount of drugs in the bloodstream will fall to virtually 0 in about 24 hours after the last dose.

4b)If drugs continue to be administered every 6 hours, the amount of drugs in the bloodstream will vary between 14.4 nanograms and 4.9 nanograms in the cycle shown in the graph.